Classification of simple weight modules for the Neveu-Schwarz algebra with a finite-dimensional weight space

Xiufu Zhang [1], Zhangsheng Xia [2]

1. School of Mathematical Sciences, Xuzhou Normal University, Xuzhou 221116, China

2. School of Sciences, Hubei University for Nationalities, Enshi 445000, China

Abstract

We show that the support of a simple weight module over the Neveu-Schwarz algebra, which has an infinite-dimensional weight space, coincides with the weight lattice and that all non-trivial weight spaces of such module are infinite-dimensional. As a corollary we obtain that every simple weight module over the Neveu-Schwarz algebra, having a non-trivial finite-dimensional weight space, is a Harish-Chandra module (and hence is either a highest or lowest weight module, or else a module of the intermediate series). This result generalizes a theorem which was originally given on the Virasoro algebra.

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1. **Introduction**

It is well known that the Virasoro algebra Vir plays a fundamental role in two-dimensional conformal quantum field theory. As the super-generalization, there are two super-Virasoro algebras called the *Neveu-Schwarz algebra* and the *Ramond algebra* corresponding to $N = \frac{1}{2}$ and $N = 1$ super-conformal field theory respectively. Let $\theta = \frac{1}{2}$ or 0 which corresponds to the Neveu-Schwarz case or the Ramond case respectively. The super-Virasoro algebra $SVir(\theta)$ is the Lie superalgebra $SVir(\theta) = SVir_{\bar{0}} \oplus SVir_{\bar{1}},$ where $SVir_{\bar{0}}$ has a basis $\{L_n, c \mid n \in \mathbb{Z}\}$ and $SVir_{\bar{1}}$ has a basis

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** Email: xfzhang@xznu.edu.cn; xzsw8577@163.com

$\{G_r \mid r \in \theta + \mathbb{Z}\}$, with the commutation relations

$$
\begin{aligned}
&[L_m, L_n] = (n-m)L_{n+m} + \delta_{m+n,0}\frac{m^3-m}{12}c,\\
&[L_m, G_r] = (r - \frac{m}{2})G_{m+r},\\
&[G_r, G_s] = 2L_{r+s} + \frac{4r^2-1}{12}\delta_{r+s,0}c,\\
&[SVir_{\bar{0}}, c] = 0 = [SVir_{\bar{1}}, c],
\end{aligned}
$$

for $m, n \in \mathbb{Z}, r, s \in \theta + \mathbb{Z}$.

The subalgebra $\mathfrak{h} = \mathbb{C}L_0 \oplus \mathbb{C}c$ is called the *Cartan subalgebra* of $SVir(\theta)$. An $\mathfrak{h}$-diagonalizable $SVir(\theta)$-module is usually called a *weight module*. If M is a weight module, then M can be written as a direct sum of its weight spaces, $M = \bigoplus M_\lambda$, where $M_\lambda = \{v \in M | L_0 v = \lambda(L_0)v, cv = \lambda(c)v\}$. We call $\{\lambda | M_\lambda \neq 0\}$ the *support* of M and is denoted by *supp*(M). A weight module is called *Harish-Chandra module* if each weight space is finite-dimensional.

In [1], [2], [3], [7], the unitary modules, the Verma modules, the Fock modules and the Harish-Chadra modules over $SVir(\theta)$ are studied. In this paper, we try to give the classification of simple weight modules for the Neveu-Schwarz algebra with a finite-dimensional weight space. Our main result generalizes a theorem which was originally given on the Virasoro algebra in [6].

For convenience, we denote the Neveu-Schwarz algebra by NS instead of $SVir(\frac{1}{2})$ and we define

$$
E_k = \begin{cases} L_k, & \text{if } \ k \in \mathbb{Z},\\ G_k, & \text{if } \ k \in \frac{1}{2} + \mathbb{Z}. \end{cases}
$$

Suppose M is a simple weight $NS-$module, then c acts on M by a scalar and M can be written as a direct sum of its weight spaces, $M = \bigoplus M_\lambda$, where $M_\lambda = \{v \in M | L_0 v = \lambda v\}$. Obviously, if $\lambda \in \mathrm{supp}(M)$, then $\mathrm{supp}(M) \subseteq \{\lambda + k | k \in \frac{\mathbb{Z}}{2}\}$, the weight lattice. Two elements $i, j \in \{\lambda + k | k \in \frac{\mathbb{Z}}{2}\}$ are called *adjacent* if $|i - j| = \frac{1}{2}$, otherwise, *unadjacent*.

In this paper, our main result is the following theorem:

Theorem 1. Let M be a simple weight $NS-$module. Assume that there exists $\lambda \in \mathbb{C}$ such that $\dim M_\lambda = \infty$. Then *supp*$(M) = \lambda + \frac{\mathbb{Z}}{2}$ and for each $k \in \frac{\mathbb{Z}}{2}$ we have $\dim(M_{\lambda+k}) = \infty$.

In [7], Y. Su proved the following result which generalizes a theorem originally given as a conjecture by Kac in [4] on the Virasoro algebra and proved by Mathieu in [5].

Theorem 2. A Harish-Chandra module over $SVir(\theta)$ is either a highest or lowest weight module, or else a module of the intermediate series.

By Theorem 1 and 2, we get the following corollaries immediately:

Corrollary 3. Let M be a simple weight $NS-$module. Assume that there exists $\lambda \in \mathbb{C}$ such that $0 < \dim M_\lambda < \infty$. Then M is a Harish-Chandra module. Consequently, M is either a highest or lowest weight module, or else a module of the intermediate series.

A weight NS-module, M, is called *mixed* module if there exist $\lambda \in \mathbb{C}$ and $k \in \frac{\mathbb{Z}}{2}$ such that $\dim M_\lambda = \infty$ and $\dim M_{\lambda+k} < \infty$.

Corrollary 4. There are no simple mixed NS-modules.

2. **Proof of the Theorem 1**

Noting that $\{G_{-\frac{3}{2}}, G_{-\frac{1}{2}}, G_{\frac{1}{2}}, G_{\frac{3}{2}}\}$ is a set of generators of NS, we have the following fact immediately:

Principal Fact: Assume that there exists $\mu \in \mathbb{C}$ and $0 \neq v \in M_\mu$, such that $G_{\frac{1}{2}}v = G_{\frac{3}{2}}v = 0$ or $G_{-\frac{1}{2}}v = G_{-\frac{3}{2}}v = 0$. Then M is a Harish-Chandra NS-module.

Lemma 1. If there are $\mu, \mu^{'} \in \{\lambda + k | k \in \frac{\mathbb{Z}}{2}\}$ such that $\dim M_\mu < \infty$ and $\dim M_{\mu^{'}} < \infty$, then $\mu, \mu^{'}$ are adjacent.

Proof. Suppose that there exist two unadjacent elements in $\{\lambda + k | k \in \frac{\mathbb{Z}}{2}\}$ correspond to finite-dimensional weight spaces or trivial vector spaces in M. Without loss of generality, we may assume that $\dim M_{\lambda+\frac{1}{2}} < \infty$ and $\dim M_{\lambda+k} < \infty$, where $k \in \{\frac{3}{2}, 2, \frac{5}{2}, 3, \cdots\}$. Let V denote the intersection of the kernels of the linear maps:

$$G_{\frac{1}{2}} : M_\lambda \to M_{\lambda+\frac{1}{2}},$$

and

$$E_k : M_\lambda \to M_{\lambda+k}.$$

Since $\dim M_\lambda = \infty, \dim M_{\lambda+\frac{1}{2}} < \infty$, we know that the kernel $kerG_{\frac{1}{2}}$ of $G_{\frac{1}{2}} : M_\lambda \to M_{\lambda+\frac{1}{2}}$ is infinite dimensional. Since $\dim M_{\lambda+k} < \infty$, we also have that the kernel of $E_k|_{kerG_{\frac{1}{2}}} : kerG_{\frac{1}{2}} \to M_{\lambda+k}$ is infinite dimensional. That is $\dim V = \infty$.

Since

$$[G_{\frac{1}{2}}, E_l] = \begin{cases} 2L_{\frac{1}{2}+l}, & \text{if } \ l \in \frac{1}{2} + \mathbb{N}, \\ (\frac{l}{2} - \frac{1}{2})G_{\frac{1}{2}+l}, & \text{if } \ l \in \mathbb{Z}_+ \setminus \{1\} \end{cases}$$

is nonzero, we get that

$$E_l V = 0, \ \forall l = \frac{1}{2}, k, k + \frac{1}{2}, k + 1, \cdots \tag{1}$$

If $k = \frac{3}{2}$ then there exists $0 \neq v \in V$ such that $G_{\frac{3}{2}}v = 0$, and M would be a Harish-Chandra module by the Principal Fact, a contradiction. Thus $k > \frac{3}{2}$ and $\dim G_{\frac{3}{2}}V = \infty$. Since $\dim M_{\lambda+\frac{1}{2}} < \infty$, there exists $0 \neq w \in G_{\frac{3}{2}}V$ such that $L_{-1}w = 0$.

Suppose $w = G_{\frac{3}{2}}u$ for some $u \in V$. For all $l \geq k > \frac{3}{2}$, using (1) we have

$$E_l w = E_l G_{\frac{3}{2}} u = \begin{cases} -G_{\frac{3}{2}} E_l u + [E_l, G_{\frac{3}{2}}]u = 0, & \text{if } \ k \in \frac{1}{2} + \mathbb{Z}_+, \\ G_{\frac{3}{2}} E_l u + [E_l, G_{\frac{3}{2}}]u = 0, & \text{if } \ k \in \mathbb{Z}_+. \end{cases}$$

Hence

$$E_l w = 0, \ \forall l = -1, k, k + \frac{1}{2}, k + 1. \cdots$$

Since $[L_{-1}, E_l] \neq 0$ for all $l \geq \frac{3}{2}$, inductively, we get

$$E_l w = 0, \ \forall l = \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots .$$

Hence M is a Harish-Chandra module by the Principal Fact. A contradiction. Then the lemma follows. $\square$

By Lemma 1, we see that there exist at most two elements in $\{\lambda + k | k \in \frac{\mathbb{Z}}{2}\}$ which correspond to finite-dimensional weight spaces or trivial vector spaces in M. Moreover, if there are two, then they are adjacent.

Lemma 2. (i) Let $0 \neq v \in M$ be such that $G_{\frac{1}{2}} v = 0$. Then

$$(\frac{1}{2} L_1 G_{\frac{1}{2}} - G_{\frac{3}{2}}) G_{\frac{3}{2}} v = 0.$$

(ii) Let $0 \neq w \in M$ be such that $G_{-\frac{1}{2}} w = 0$. Then

$$(\frac{1}{2} L_{-1} G_{-\frac{1}{2}} + G_{-\frac{3}{2}}) G_{-\frac{3}{2}} w = 0.$$

Proof. Note that $L_1 v = G_{\frac{1}{2}} G_{\frac{1}{2}} v = 0, L_{-1} w = G_{-\frac{1}{2}} G_{-\frac{1}{2}} w = 0$, we can easily check by a direct calculation that Lemma 2 holds. $\square$

Lemma 3. Let M be a simple weight NS-module satisfying $\dim M_\mu < \infty$ and $\dim M_{\mu + \frac{1}{2}} < \infty$. Then $\mu \in \{-1, \frac{1}{2}\}$.

Proof. Let V be the kernel of $G_{\frac{1}{2}} : M_{\mu - \frac{1}{2}} \to M_\mu$. Since $\dim M_{\mu - \frac{1}{2}} = \infty$ and $\dim M_\mu < \infty$, we see that $\dim V = \infty$. For any $0 \neq v \in V$, consider the element $G_{\frac{3}{2}} v$. By the Principal Fact, $G_{\frac{3}{2}} v = 0$ would imply that M is a Harish-Chandra module, a contradiction. Hence $G_{\frac{3}{2}} v \neq 0$, in particular, $\dim G_{\frac{3}{2}} V = \infty$. This implies that there exists $w \in G_{\frac{3}{2}} V$ such that $w \neq 0$ and $G_{-\frac{1}{2}} w = 0$ since $\dim M_{\mu + \frac{1}{2}} < \infty$. From Lemma 2 we have $(\frac{1}{2} L_1 G_{\frac{1}{2}} - G_{\frac{3}{2}}) w = 0$. In particular, we have $L_{-1} G_{-\frac{1}{2}} (\frac{1}{2} L_1 G_{\frac{1}{2}} - G_{\frac{3}{2}}) w = 0$. By a direct calculation we obtain

$$L_{-1} G_{-\frac{1}{2}} (\frac{1}{2} L_1 G_{\frac{1}{2}} - G_{\frac{3}{2}}) \equiv 2L_0^2 - 3L_0 \bmod U(NS) G_{-\frac{1}{2}}.$$

But $w \in M_{\mu+1}$, which means $L_0 w = (\mu+1)w$ and hence

$$2(\mu+1)^2 - 3(\mu+1) = 0.$$

So $\mu \in \{-1, \frac{1}{2}\}$. $\square$

From Lemma 3 we know that if M has two finite dimension weight spaces, then they must be M_{-1}, $M_{-\frac{1}{2}}$ or $M_{\frac{1}{2}}, M_1$.

Lemma 4. $\mu \neq -1$ and $\mu \neq \frac{1}{2}$.

Proof. By the symmetry, we need only to prove that $\mu \neq \frac{1}{2}$. Let V be the kernel of $G_{\frac{1}{2}} : M_0 \to M_{\frac{1}{2}}$. Then $\dim V = \infty$. For $v \in V$, using $G_{\frac{1}{2}} v = L_0 v = 0$, we have

$$G_{\frac{1}{2}} G_{-\frac{1}{2}} v = 2L_0 v - G_{-\frac{1}{2}} G_{\frac{1}{2}} v = 0. \tag{2}$$

If $G_{-\frac{1}{2}} V$ were infinite-dimensional, there would exists $0 \neq w \in G_{-\frac{1}{2}} V$ such that $G_{\frac{1}{2}} w = 0$ (by (2)) and $G_{\frac{3}{2}} w = 0$ (since $\dim M_1 < \infty$). Then the Principal Fact would then imply that M is a Harish-Chandra module, a contradiction. Hence

$$\dim G_{-\frac{1}{2}} V < \infty.$$

This means that the kernel W of the linear map

$$G_{-\frac{1}{2}} : V \to M_{-\frac{1}{2}}$$

is infinite-dimensional. For every $x \in W$ we have

$$G_{\frac{1}{2}} G_{-\frac{3}{2}} x = 2L_{-1} x - G_{-\frac{3}{2}} G_{\frac{1}{2}} x = 0. \tag{3}$$

If there exists $0 \neq x \in W$ such that $G_{-\frac{3}{2}} x = 0$, then we would have $G_{-\frac{3}{2}} x = G_{-\frac{1}{2}} x = 0$ and the Principal Fact would imply that M is a Harish-Chandra module, a contradiction. Thus

$$\dim G_{-\frac{3}{2}} W = \infty.$$

Let H denote the kernel of the linear map $L_2 : G_{-\frac{3}{2}} W \to M_{\frac{1}{2}}$. Since $\dim G_{-\frac{3}{2}} W = \infty$ and $\dim M_{\frac{1}{2}} < \infty$, we have $\dim H = \infty$. For every $y \in H$, we also have $G_{\frac{1}{2}} y = 0$ by (3), implying by induction that $E_k H = 0$ for all $k = \frac{1}{2}, 1, 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \cdots$. If $G_{\frac{3}{2}} h = 0$ for some $0 \neq h \in H$, then the Principal Fact implies that M is a Harish-Chandra module, a contradiction. Hence

$$\dim G_{\frac{3}{2}} H = \infty.$$

For every $h \in H$ and $k \geq 2$, we have

$$E_k G_{\frac{3}{2}} h = \begin{cases} 2L_{\frac{3}{2}+k} h - G_{\frac{3}{2}} G_k h = 0, & \text{if } \ k \in \frac{1}{2} + \mathbb{Z}_+, \\ (\frac{3}{2} - \frac{k}{2}) G_{\frac{3}{2}+k} h + G_{\frac{3}{2}} L_k h = 0, & \text{if } \ k \in \mathbb{Z}_+. \end{cases}$$

Hence

$$E_k G_{\frac{3}{2}} h = 0,\ \forall k = 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}, \cdots. \tag{4}$$

Let, finally, K denote the infinite-dimensional kernel of the linear map

$$G_{\frac{1}{2}} : G_{\frac{3}{2}} H \to M_{\frac{1}{2}}.$$

If $G_{\frac{3}{2}} z = 0$ for some $0 \neq z \in K$, then the Principal Fact implies that M is a Harish-Chandra module, a contradiction. Hence $G_{\frac{3}{2}} z \neq 0, \forall 0 \neq z \in K$. For every $z \in K$ and $k \geq 2$, by (4), we have $\dim G_{\frac{3}{2}} K = \infty$ and

$$E_k G_{\frac{3}{2}} z = \begin{cases} 2L_{\frac{3}{2}+k} z - G_{\frac{3}{2}} G_k z = 0, & \text{if } \ k \in \frac{1}{2} + \mathbb{Z}_+, \\ (\frac{3}{2} - \frac{k}{2}) G_{\frac{3}{2}+k} z + G_{\frac{3}{2}} L_k z = 0. & \text{if } \ k \in \mathbb{Z}_+. \end{cases}$$

Hence $E_k G_{\frac{3}{2}} K = 0$ for all $k \geq 2$. At the same time, since $\dim G_{\frac{3}{2}} K = \infty$ and $\dim M_1 < \infty$, we can choose some $0 \neq t \in G_{\frac{3}{2}} K$ such that $G_{-\frac{1}{2}} t = 0$, by induction, we get $E_i t = 0$ for all $i > 0$ and thus M is a Harish-Chandra module by the Principal Fact. This last contradiction completes the proof of lemma 4. □

By lemma 1, Lemma 3 and Lemma 4, we get the following lemma immediately:

Lemma 5. There is at most one element $\mu \in \{\lambda + k | k \in \frac{\mathbb{Z}}{2}\}$ such that $\dim M_\mu < \infty$.

Now the proof of Theorem 1 follows from the following Lemma:

Lemma 6. There is no $\mu \in \{\lambda + k | k \in \frac{\mathbb{Z}}{2}\}$ such that $\dim M_\mu < \infty$.

Proof. Suppose that $\dim M_\mu < \infty$ and $\dim M_\nu = \infty$ for all $\mu \neq \nu \in \{\lambda + k | k \in \frac{\mathbb{Z}}{2}\}$. Define

$$V = ker(G_{\frac{1}{2}} : M_{\mu-\frac{1}{2}} \to M_\mu) \cap ker(G_{\frac{1}{2}} G_{-\frac{3}{2}} G_{\frac{3}{2}} : M_{\mu-\frac{1}{2}} \to M_\mu) \cap ker(L_{-1} G_{\frac{3}{2}} : M_{\mu-\frac{1}{2}} \to M_\mu),$$

$$W = ker(G_{-\frac{1}{2}} : M_{\mu+\frac{1}{2}} \to M_\mu) \cap ker(G_{-\frac{1}{2}} G_{\frac{3}{2}} G_{-\frac{3}{2}} : M_{\mu-\frac{1}{2}} \to M_\mu) \cap ker(L_1 G_{-\frac{3}{2}} : M_{\mu-\frac{1}{2}} \to M_\mu).$$

Since $\dim M_{\mu-\frac{1}{2}} = \infty$ and $\dim M_\mu < \infty$, V is a vector subspace of finite codimension in $M_{\mu-\frac{1}{2}}$. Since $\dim M_{\mu+\frac{1}{2}} = \infty$ and $\dim M_\mu < \infty$, W is a vector subspace of finite codimension in $M_{\mu+\frac{1}{2}}$. In order not to get a direct contradiction using the Principal Fact, we assume that $G_{\frac{3}{2}} v \neq 0$ for all $0 \neq v \in V$ and $G_{-\frac{3}{2}} w \neq 0$ for all $0 \neq w \in W$. Then $\dim G_{\frac{3}{2}} V = \infty$ and $\dim G_{-\frac{3}{2}} W = \infty$.

Claim $\mu \neq \pm 1$. Moreover, for any $v \in V, w \in W$, we have

$$G_{\frac{1}{2}} G_{-\frac{1}{2}} G_{\frac{3}{2}} v = \tau G_{\frac{3}{2}} v, \tag{5}$$

$$G_{-\frac{1}{2}} G_{\frac{1}{2}} G_{-\frac{3}{2}} w = \tau' G_{-\frac{3}{2}} w, \tag{6}$$

where $\tau = \frac{(\mu+1)(2\mu-1)}{\mu-1}$ and $\tau' = \frac{(\mu-1)(2\mu+1)}{\mu+1}$.

Proof of the Claim. It can be checked directly that

$$L_{-1}G_{-\frac{1}{2}}(\frac{1}{2}L_1G_{\frac{1}{2}} - G_{\frac{3}{2}}) \equiv 2L_0^2 - 3L_0 - L_0G_{\frac{1}{2}}G_{-\frac{1}{2}} + 2G_{\frac{1}{2}}G_{-\frac{1}{2}} \bmod U(NS)L_{-1}, \quad (7)$$

and

$$L_1G_{\frac{1}{2}}(\frac{1}{2}L_{-1}G_{-\frac{1}{2}} + G_{-\frac{3}{2}}) \equiv -2L_0^2 - 3L_0 + L_0G_{-\frac{1}{2}}G_{\frac{1}{2}} + 2G_{-\frac{1}{2}}G_{\frac{1}{2}} \bmod U(NS)L_1. \quad (8)$$

For any $0 \neq v \in V$, by Lemma 2(i) and (7), we have

$$2L_0^2G_{\frac{3}{2}}v - 3L_0G_{\frac{3}{2}}v - L_0G_{\frac{1}{2}}G_{-\frac{1}{2}}G_{\frac{3}{2}}v + 2G_{\frac{1}{2}}G_{-\frac{1}{2}}G_{\frac{3}{2}}v = 0.$$

Then

$$(2(\mu+1)^2 - 3(\mu+1))G_{\frac{3}{2}}v - (\mu-1)G_{\frac{1}{2}}G_{-\frac{1}{2}}G_{\frac{3}{2}}v = 0.$$

If $\mu = 1$, then $G_{\frac{3}{2}}v = 0$, a contradiction. Thus $\mu \neq 1$, and (4) holds.

For any $0 \neq w \in W$, by Lemma 2(ii) and (8), we have

$$-2L_0^2G_{-\frac{3}{2}}w - 3L_0G_{-\frac{3}{2}}w + L_0G_{-\frac{1}{2}}G_{\frac{1}{2}}G_{-\frac{3}{2}}w + 2G_{-\frac{1}{2}}G_{\frac{1}{2}}G_{-\frac{3}{2}}w = 0.$$

Then

$$(2(\mu-1)^2 + 3(\mu-1))G_{-\frac{3}{2}}w - (\mu+1)G_{-\frac{1}{2}}G_{\frac{1}{2}}G_{-\frac{3}{2}}w = 0.$$

If $\mu = -1$, then $G_{-\frac{3}{2}}w = 0$, a contradiction. Thus $\mu \neq -1$, and (6) holds. This completes the proof of the Claim.

By (4) and (6), we see that $G_{\frac{1}{2}}G_{-\frac{1}{2}}G_{\frac{3}{2}}v = 0$ if and only if $\mu = \frac{1}{2}$, $G_{-\frac{1}{2}}G_{\frac{1}{2}}G_{-\frac{3}{2}}w = 0$ if and only if $\mu = -\frac{1}{2}$. There only two cases may occur: $\mu \notin \{1, -1, \frac{1}{2}\}$ or $\mu \notin \{1, -1, -\frac{1}{2}\}$. Because of the symmetry of our situation, to complete the proof of the Theorem, it is enough to show that a contradiction can be derived when $\mu \notin \{1, -1, \frac{1}{2}\}$.

Suppose $\mu \notin \{1, -1, \frac{1}{2}\}$, then $\tau \neq 0$ and $G_{-\frac{1}{2}}G_{\frac{3}{2}}v \neq 0$ for any $0 \neq v \in V$. Set

$$S = G_{-\frac{1}{2}}G_{\frac{3}{2}}V \cap W.$$

Since both $G_{-\frac{1}{2}}G_{\frac{3}{2}}V$ and W have finite codimension in $M_{\mu+\frac{1}{2}}$, we have $\dim S = \infty$.

Note that

$$G_{-\frac{3}{2}}(\frac{1}{2}L_1G_{\frac{1}{2}} - G_{\frac{3}{2}}) \equiv -G_{\frac{1}{2}}G_{-\frac{1}{2}} - \frac{1}{2}L_1G_{\frac{1}{2}}G_{-\frac{3}{2}} + G_{\frac{3}{2}}G_{-\frac{3}{2}} - \frac{2}{3}c \bmod U(NS)L_{-1}.$$

For any $v \in V$, by Lemma 2(i), (4) and the definition of V, we have

$$G_{\frac{3}{2}}G_{-\frac{3}{2}}G_{\frac{3}{2}}v = G_{\frac{1}{2}}G_{-\frac{1}{2}}G_{\frac{3}{2}}v + \frac{2}{3}cG_{\frac{3}{2}}v = pG_{\frac{3}{2}}v, \text{ for some } p \in \mathbb{C}. \quad (9)$$

Similarly, For any $w \in W$, by

$$G_{\frac{3}{2}}(\frac{1}{2}L_{-1}G_{-\frac{1}{2}} + G_{-\frac{3}{2}}) \equiv G_{-\frac{1}{2}}G_{\frac{1}{2}} - \frac{1}{2}L_{-1}G_{-\frac{1}{2}}G_{\frac{3}{2}} - G_{-\frac{3}{2}}G_{\frac{3}{2}} + \frac{2}{3}c \bmod U(NS)L_1,$$

Lemma 2(ii) , (6) and the definition of W, we have

$$G_{-\frac{3}{2}}G_{\frac{3}{2}}G_{-\frac{3}{2}}w = G_{-\frac{1}{2}}G_{\frac{1}{2}}G_{-\frac{3}{2}}w + \frac{2}{3}cG_{-\frac{3}{2}}w = qG_{-\frac{3}{2}}w, \text{ for some } q \in \mathbb{C}. \tag{10}$$

Choose $v \in V$ such that $0 \neq G_{-\frac{1}{2}}G_{\frac{3}{2}}v \in S$. Set

$$x = G_{\frac{3}{2}}v, y = G_{-\frac{1}{2}}x, h = G_{-\frac{3}{2}}x, z = G_{-\frac{3}{2}}y. \tag{11}$$

By the definitions of W, S and (10), we have

$$G_{-\frac{1}{2}}(G_{\frac{3}{2}}G_{-\frac{3}{2}}y - qy) = 0 - 0 = 0$$

and

$$G_{-\frac{3}{2}}(G_{\frac{3}{2}}G_{-\frac{3}{2}}y - qy) = G_{-\frac{3}{2}}G_{\frac{3}{2}}G_{-\frac{3}{2}}y - qG_{-\frac{3}{2}}y = 0.$$

So $G_{\frac{3}{2}}G_{-\frac{3}{2}}y - qy = 0$ by the Principal Fact, and we get the formula:

$$G_{\frac{3}{2}}z = qy. \tag{12}$$

Moreover, we have

$$G_{\frac{1}{2}}z = G_{\frac{1}{2}}G_{-\frac{3}{2}}y = (2L_{-1} - G_{-\frac{3}{2}}G_{\frac{1}{2}})y = 2G^2_{-\frac{1}{2}}y - G_{-\frac{3}{2}}G_{\frac{1}{2}}G_{-\frac{1}{2}}G_{\frac{3}{2}}v = -\tau G_{-\frac{3}{2}}x = -\tau h,$$

i.e.,

$$G_{\frac{1}{2}}z = -\tau h. \tag{13}$$

Let U_+ and U_- denote the subalgebras of $U(NS)$, generated by $G_{\frac{1}{2}}, G_{\frac{3}{2}}$ and $G_{-\frac{1}{2}}, G_{-\frac{3}{2}}$, respectively. We want to prove that the following vector space

$$N = U_+x \oplus U_+y \oplus U_-z \oplus U_-h$$

is a proper weight submodule of M, which will derive a contradiction, as desired.

Since x, y, z and h are all eigenvectors for L_0, we see that N decomposes into a direct sum of weight spaces which are obviously finite-dimensional. It remains to show that N is stable under the action of the following four operators: $G_{\frac{1}{2}}, G_{\frac{3}{2}}, G_{-\frac{1}{2}}$ and $G_{-\frac{3}{2}}$. That $G_{\frac{i}{2}}U_+x \subset U_+x$, $G_{\frac{i}{2}}U_+y \subset U_+y$, $G_{-\frac{i}{2}}U_-h \subset U_-h$ and $G_{-\frac{i}{2}}U_-z \subset U_-z$ is clear for $i = 1, 3$.

For any $a \in U_+, a' \in U_-$, there exists $a_{i,j}, b_{i,j}, c_{i,j} \in U_+$ and $a'_{i,j}, b'_{i,j}, c'_{i,j} \in U_-$ such that

$$G_{-\frac{1}{2}}a = aG_{-\frac{1}{2}} + \sum_{i,j} a_{i,j}L_0^i c^j,$$

$$G_{-\frac{3}{2}}a = aG_{-\frac{3}{2}} + \sum_{i,j} a_{i,j}L_0^i c^j + \sum_{i,j} b_{i,j}L_0^i c^j L_{-1} + \sum_{i,j} c_{i,j}L_0^i c^j G_{-\frac{1}{2}},$$

$$G_{\frac{1}{2}}a' = a'G_{\frac{1}{2}} + \sum_{i,j} a'_{i,j}L_0^i c^j,$$

$$G_{\frac{3}{2}}a' = a'G_{\frac{3}{2}} + \sum_{i,j} a'_{i,j} L_0^i c^j + \sum_{i,j} b'_{i,j} L_0^i c^j L_1 + \sum_{i,j} c'_{i,j} L_0^i c^j G_{\frac{1}{2}}.$$

Thus, to show $G_{-\frac{i}{2}}U_+x, G_{-\frac{i}{2}}U_+y, G_{\frac{i}{2}}U_-h, G_{\frac{i}{2}}U_-z \subset N$, we need only to show that $G_{-\frac{i}{2}}x, G_{-\frac{i}{2}}y, G_{\frac{i}{2}}h, G_{\frac{i}{2}}z, G_{-\frac{i}{2}}G_{\frac{j}{2}}x, G_{-\frac{i}{2}}G_{\frac{j}{2}}y, G_{\frac{i}{2}}G_{-\frac{j}{2}}h, G_{\frac{i}{2}}G_{-\frac{j}{2}}z \in N$ for $i, j = 1, 3$.

Now we can check the following formulas one by one via the definitions of V, W, S and (5)-(13):

$$G_{-\frac{1}{2}}x = y,$$

$$G_{-\frac{1}{2}}y = 0,$$

$$G_{-\frac{3}{2}}x = h,$$

$$G_{-\frac{3}{2}}y = z,$$

$$G_{\frac{1}{2}}h = G_{\frac{1}{2}}G_{-\frac{3}{2}}G_{\frac{3}{2}}v = 0,$$

$$G_{\frac{1}{2}}z = -\tau h,$$

$$G_{\frac{3}{2}}h = G_{\frac{3}{2}}G_{-\frac{3}{2}}G_{\frac{3}{2}}v = pG_{\frac{3}{2}}v = px,$$

$$G_{\frac{3}{2}}z = qy,$$

$$G_{\frac{1}{2}}G_{-\frac{3}{2}}x = 0,$$

$$G_{\frac{3}{2}}G_{-\frac{3}{2}}x = px,$$

$$G_{\frac{1}{2}}G_{-\frac{3}{2}}y = -\tau h,$$

$$G_{\frac{3}{2}}G_{-\frac{3}{2}}y = qy,$$

$$G_{\frac{1}{2}}G_{\frac{1}{2}}G_{-\frac{3}{2}}y = 0,$$

$$G_{\frac{3}{2}}G_{\frac{1}{2}}G_{-\frac{3}{2}}y = -\tau px,$$

$$L_{-1}x = G^2_{-\frac{1}{2}}x = 0,$$

$$L_{-1}y = G^2_{-\frac{1}{2}}y = 0,$$

$$G_{-\frac{1}{2}}G_{\frac{3}{2}}h = py,$$

$$G_{-\frac{3}{2}}G_{\frac{3}{2}}h = ph,$$

$$G_{-\frac{1}{2}}G_{\frac{3}{2}}z = 0,$$

$$G_{-\frac{3}{2}}G_{\frac{3}{2}}z = qz,$$

$$G_{-\frac{1}{2}}G_{-\frac{1}{2}}G_{\frac{3}{2}}h = 0,$$

$$G_{-\frac{3}{2}}G_{-\frac{1}{2}}G_{\frac{3}{2}}h = pz,$$

$$L_1 h = G^2_{\frac{1}{2}}h = 0,$$

$$L_1 z = G^2_{\frac{1}{2}}z = 0.$$

This completes the proof of Lemma 6 and then of Theorem 1. $\square$

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